JOURNAL OF FUNCTIONAL ANALYSIS 98, 169–193 (1991)

Type Decomposition for von Neumann Algebra Embeddings

VICTOR KAFTAL*

Department of Mathematical Sciences, University of Cincinnati, Cincinnati, Ohio 45221-0025

Communicated by C. Foias

Received January 14, 1990

This paper establishes a type decomposition theory for embeddings of a von Neumann algebra A into a von Neumann algebra M, by studying projections in A that are semifinite (resp. abelian) relative to M. An application of this theory is a complete characterization of the subalgebras A of M that have an n-equipartition of the identity, i.e., $I = \sum_{k=1}^{n} p_k$ with projections $p_k \in A$ which are equivalent in M. From this follows R. V. Kadison's result (*Amer. J. Math.* **106**, 1984, 1451–1468) that masas of M always have equipartitions and hence that normal operator valued matrices are diagonalizable. A second application is an answer to a question by G. K. Pedersen and E. Størmer (*Indiana Univ. Math. J.* **23**, 1973, 121–129) on the characterization of $\sim_G$-finite projections in discrete crossed products. © 1991 Academic Press, Inc.

1. INTRODUCTION

Kadison has proven in [3] that every normal matrix $x \in M = R \otimes M_n(\mathbb{C})$ with entries in a σ-finite von Neumann algebra R is diagonalizable. Equivalently (see [3, Theorem 3.18]), each masa A of M (maximal abelian selfadjoint subalgebra) contains n mutually orthogonal equivalent projections with sum I (an n-equipartition of I in the notation of our Definition 10).

In the course of the proof, Kadison started a relative type decomposition for the embedding of masas into von Neumann algebras and proposed that such a theory be carried out for more general subalgebras.

Our goal is to establish a type decomposition theory for a general embedding of a von Neumann algebra A into a von Neumann algebra M, and then apply it to completely characterize the subalgebras of M that have an n-equipartition of I (Theorems 25 and 29). As a special case (Corollary 31), we derive Kadison's result [3, Theorem 3.18] for masas.

* Partially supported by a grant by the Taft Foundation.

0022-1236/91 $3.00

A second application is to discrete crossed products: we derive a result by Størmer [11, Lemma 9] on the existence of invariant traces (Corollary 35), and we settle in the affirmative (Corollary 37) a question by Pedersen and Størmer [8] on the characterization of $\sim_G$-finite projections.

In both applications, we make the (essential) assumption that the center Z of M is contained in A, and in following the lead of Kadison [3] and Pedersen and Størmer [8], we avoid (nontrivial) complications connected with higher cardinals, by assuming that the algebra M is σ-finite (i.e., countably decomposable).

The definitions in our theory are modelled on the intrinsic (or absolute) type decomposition of von Neumann algebras:

An M-semifinite projection of A is the sum of projections in A finite relative to M, and thus an M-type III projection is one that majorizes no nonzero projections of A finite relative to M.

An M-abelian projection is a projection $p \in A$ such that $A_p = Z_p$, where Z denotes the center of M, an M-discrete projection is the sum of mutually orthogonal M-abelian projections, and an M-continuous projection is one that majorizes no nonzero M-abelian projections.

The algebra A decomposes canonically (along the center Z_A of A) into the sum of an M-semifinite and an M-type III part, and also into the sum of an M-discrete and an M-continuous part (Theorems 3 and 9). The inclusions among these parts that are familiar in the intrinsic type theory, no longer hold for the embedding case: e.g., there are M-discrete algebras that are also M-type III, and there are M-continuous algebras that are finite (relative to M).

Essential for our key Theorem 25, is the canonical decomposition (along the center Z of M) of the M-discrete part of A into M-type I_n homogeneous parts, where the identity decomposes into n mutually orthogonal M-abelian projections with the same central support (in Z) (Theorem 15, Proposition 19).

Some of our ideas and definitions are inspired by Kadison's paper [3]. Another debt that we acknowledge is to Størmer's theory in the setting of discrete crossed products in [11]. The notion and some of the properties of M-abelian projections have also appeared in [1, 2], with somewhat different definitions and names.

This paper was initiated when the author was visiting at the University of Pennsylvania. He thanks Richard Kadison for his very warm hospitality and for invaluable stimulus for this research. The author is grateful also to his colleagues at the University of Texas A&M and at the University of Cincinnati who listened patiently to rough versions of this paper and suggested many improvements. A special thanks goes to Gary Weiss for his help with the proof of Proposition 19.

2. Relative Semifinite Projections

Definition 1. A projection $p \in A$ is semifinite relative to the embedding of A into M (M-semifinite for short) if every nonzero subprojection $q \in A$ of p majorizes a finite nonzero projection $r \in A$.

A projection $p \in A$ is purely infinite relative to the embedding of A into M (or M-type III for short) if p majorizes no nonzero finite projections of A.

A is M-semifinite (resp. M-type III) if the identity I is M-semifinite (resp. M-type III).

In this paper, unless A is explicitly mentioned, finite, equivalent, etc., will always mean relative to M. Similarly, center and central support $c(p)$ of a projection p will always refer to the center Z of M. When the center of A is meant, we shall use the notations Z_A, $c_A(p)$, etc. H will denote a separable Hilbert space, infinite unless otherwise stated.

Lemma 2. *Let $p \in A$ be a projection. Then the following conditions are equivalent*:

(i) *p is M-semifinite.*

(ii) *p is the sum of mutually orthogonal finite projections of A.*

(iii) *p is the supremum of a family of finite projections of A.*

(iv) *There is a net $x_\gamma \in J(M) \cap A$ converging σ-weakly to p, where $J(M)$ is the norm closed two sided ideal of M generated by the finite projections of M.*

Proof. (i) $\Rightarrow$ (ii) follows from a routine maximality argument and (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious. Assume now that $p \neq 0$ and that there is a net $x_\gamma \in J(M) \cap A$ converging σ-weakly to p. Let $0 \neq q \leqslant p$ be a projection in A, then we can find an index γ such that $qx_\gamma x_\gamma^* q \neq 0$, and a number $\alpha > 0$ such that $r = \chi[\alpha, \infty)(qx_\gamma x_\gamma^* q) \neq 0$ (i.e., r is the spectral projection of $qx_\gamma x_\gamma^* q$ corresponding to the interval $[\alpha, \infty)$). Clearly, $r \in A$, $r \leqslant q$ and since $r \leqslant \alpha^{-1} qx_\gamma x_\gamma^* q$ and $J(M) \cap A$ is an ideal of A, we see that r is finite, which proves the implication (iv) $\Rightarrow$ (i). Q.E.D.

Notice that if p is M-semifinite, then it is also semifinite both relative to the algebra M and relative to the algebra A. In general both the reverse implications as well as the implications between semifiniteness relative to M and semifiniteness relative to A are false. Indeed any nonzero projection in a finite subalgebra A of a type III algebra M is finite relative to A, but it can be neither semifinite relative to M, nor M-semifinite. Similarly, any nonzero projection in a type III subalgebra A of a semifinite algebra M is semifinite relative to M, but it is neither semifinite relative to A, nor it is M-semifinite. A more illuminating example is given by the continuous masa

A of $M = B(H)$, where all the projections are semifinite relative to M and finite relative to A, but are not M-semifinite.

Notice that every subprojection of an M-semifinite (resp. M-type III) projection is also M-semifinite (resp. M-type III) and that the only projection that is at the same time M-semifinite and M-type III is zero. Thus, reasoning as in the proof of the canonical decomposition of a von Neumann algebra into a semifinite and a purely infinite part, we obtain:

THEOREM 3. *There is a unique decomposition of the identity into two mutually orthogonal projections e_s, $e_\infty \in Z_A$, such that e_s is M-semifinite and e_∞ is M-type III.*

Proof. The uniqueness is clear by the above remarks. Let

$$e_s = \sup\{p \in A \mid p \text{ finite projection}\}.$$

Then $e_\infty = I - e_s$ is M-type III by definition, and e_s is M-semifinite by Lemma 2(iii). For every unitary operator $u \in A$, we have

$$ue_s u^* = \sup\{upu^* \mid p \in A, \text{ finite projection}\} = e_s.$$

Thus $e_s \in Z_A$. Q.E.D.

Remark 4. Since $\overline{J(M) \cap A}^{\sigma w}$ is a σ-weakly closed two-sided ideal of A, there is a projection $e \in Z_A$ such that $\overline{J(M) \cap A}^{\sigma w} = eA$ [5, Theorem 6.8.8]. By using Lemma 2(iv), it is easy to verify that $e = e_S$

PROPOSITION 5. *Let p be an M-semifinite projection, then there is a subprojection $q \in A$ of p which is finite relative to A and such that $c_A(q) = c_A(p)$.*

Proof. We can assume that $p \neq 0$ and let q be the sum of a maximal family of nonzero finite subprojections $q_\gamma \in A$ of p having mutually orthogonal Z_A-central supports $c_A(q_\gamma)$. Then q is a subprojection of p and it is finite relative to A since each q_γ is finite (relative to M) and hence finite relative to A. Assume by contradiction that $f = c_A(p) - c_A(q) \neq 0$. Then $pf \in A$ and $0 \neq pf \leqslant p$, thus pf majorizes some nonzero finite projection $r \in A$, against the maximality of the family $\{q_\gamma\}$. Q.E.D.

Notice that since the central supports of the projections q_γ are not necessarily orthogonal, the projection q needs not to be finite (relative to M). Consider for instance the case when A is the atomic masa of $B(H)$ and $p = 1$.

Since a projection can be finite relative to A, but may fail to be M-semifinite, we see that the condition in Proposition 5 is not sufficient.

Notice also that the analogous statement, with the condition that q is finite relative to M and $c(q)=c(p)$, holds if $Z \subset A$ (with a similar proof based on a maximality argument), but fails to be true in general: indeed consider $M=B(H)\oplus C$, C the atomic masa of $B(H)$, and $A=\{c\oplus c \mid c\in C\}$. Then clearly A is M-semifinite, but the only projection of A that has central support (in M) equal to the identity, is the identity itself.

3. Relative Abelian Projections

A projection p in a von Neumann algebra M is said to be abelian if it is minimal among the projections having the same central support, or, equivalently, if $M_p=Z_p$ (M_p and Z_p are the restriction to the range of p of the algebras pMp and Zp, respectively). The notion of abelian projections and the ensuing notions of discrete and continuous projections have been generalized to various settings, see [1, Definition, Sect. 6; 2, Definition 3; 11, Definition 2]. In the case of the embedding of A into M we generalize these notions as follows:

Definition 6. Let $p\in A$ be a projection, then:

p is abelian relative to the embedding of A into M (M-abelian for short) if $A_p=Z_p$;

p is discrete relative to the embedding of A into M (M-discrete for short) if every nonzero subprojection $q\in A$ of p majorizes a nonzero M-abelian projection $r\in A$;

p is continuous relative to the embedding of A into M (M-continuous for short) if p majorizes no nonzero M-abelian projections.

A is M-discrete (resp. M-continuous) if the identity I is M-discrete (resp. M-continuous).

Thus for a projection $p\in A$ there are three notions of abelianess: relative to M, relative to the embedding of A into M, and relative to A.

If the center Z of M is contained in M, then the following implications hold;

$$p \text{ is abelian relative to } M \Rightarrow p \text{ is } M\text{-abelian} \Rightarrow p \text{ is abelian relative to } A,$$

and we see that the reverse implications are in general false by considering the case when M is a factor and hence a projection is abelian relative to M (resp. it is M-abelian) if and only if it is minimal in M (resp. in A).

If Z is not contained in A, then the second implication is still true (and also the first statement implies the third), but the first implication may fail to hold, as we see by considering the projection $p=I$ and the embedding of $A=\mathbb{C}I$ into an abelian algebra $M\neq A$.

The first implication can always be reversed when A is a masa of M (i.e., $A=A'\cap M$). More generally:

LEMMA 7. *Let $p\in A$ be a projection and assume that A_p is a masa of M_p. Then p is M-abelian if and only if p is abelian relative to M.*

Proof. Assume that $M_p\neq Z_p=A_p$, then there is a selfadjoint element $x\in A_p$ with $x\notin Z_p$. But then the von Neumann subalgebra of M_p generated by x and Z_p is abelian and properly contains $Z_p=A_p$. Q.E.D.

Notice that by definition, zero is the only projection that is at the same time M-discrete and M-continuous. Also, the notions of M-discreteness and M-continuity are hereditary, i.e., a subprojection of an M-discrete (resp. M-continuous) projection is M-discrete (resp. M-continuous). In the next lemma we shall see that the same holds for the notion of M-abelianess.

Many (but not all) of the usual properties of abelian, discrete, and continuous projections carry over to the case of the embedding of A into M. We collect in Lemma 8 several of the properties that we shall use more often.

LEMMA 8. *Let p be a projection in A.*

(i) *p is M-abelian if and only if $pZ\subset A$ and for every subprojection $q\in A$ of p it follows that $q=pc(q)$.*

(ii) *If p is M-abelian and $q\precsim_A p$, then q is M-abelian.*

(iii) *If p is M-discrete (resp. M-continuous) and $q\precsim_A p$, then q is M-discrete (resp. M-continuous).*

(iv) *If p is the supremum of M-discrete projections, then p is M-discrete.*

(v) *p is M-discrete if and only if it is the sum of mutually orthogonal M-abelian projections.*

(vi) *If p is M-discrete, then $pZ\subset A$.*

(vii) *If p is the sum of a family of M-abelian projections with mutually orthogonal central supports, then p is M-abelian.*

(viii) *If $p\leqslant q\in M$, then p is M-abelian (resp. M-discrete, M-continuous) if and only if it is abelian (resp. discrete, continuous) relative to the embedding of A_q into M_q.*

Proof. (i) Assume that p is M-abelian, then for every $z\in Z$ there is an element $a\in A$ such that $pz=pzp=pap$, so that $pz\in A$. Let $q\in A$ be a subprojection of p, then $q=pz$ for some $z\in Z$, hence $q=pzz^*c(q)$. By considering the spectral resolution of $zz^*c(q)$, we can easily show that $zz^*c(q) = c(q)$. Conversely, assume that for every subprojection $q \in A$ of p it follows that $q = pc(q)$; then $A_p \subset Z_p$. If furthermore $pZ \subset A$, then $Z_p \subset A_p$.

(ii) Assume first that $q \leqslant p$, then by (i), $q = pc(q)$, and hence

$$A_q = (A_p)_{c(q)} = (Z_p)_{c(q)} = Z_q.$$

Assume now that $q \sim_A p$, and let $u \in A$ be a partial isometry implementing the equivalence, so that $q = upu^*$. Since p is M-abelian, for every $a \in A$ there is a $z \in Z$ for which $pu^*aup = pz$, so that

$$qaq = upu^*aupu^* = upzu^* = qz.$$

Thus q is M-abelian too.

(iii) Follows immediately from the definition and from (ii).

(iv) Assume that p is the supremum of a family $\{p_\gamma\}$ of M-discrete projections. Then for every nonzero suprojection $q \in A$ of p there is an index γ such that $qp_\gamma \neq 0$ and hence a number $\alpha > 0$ such that the spectral projection $r = \chi[\alpha, \infty)(p_\gamma q p_\gamma) \neq 0$. Since $r \leqslant p_\gamma$, we can find an M-abelian projection $0 \neq r' \leqslant r$. Let $u \in A$ be the partial isometry in the polar decomposition of qp_γ, then

$$0 \neq r'' = ur'u^* \leqslant up_\gamma q p_\gamma u^* = qp_\gamma q \leqslant q.$$

By (ii), r'' is M-abelian and hence p is M-discrete.

(v) A routine maximality argument yields the necessity of the condition in (v). The sufficiency part follows from (iv) because M-abelian projections are M-discrete by (ii).

(vi) Follows immediately from (v) and (i).

(vii) Let p be the sum of the family $\{p_\gamma\}$ of M-abelian projections with mutually orthogonal central supports $c(p_\gamma)$. By (iv), p is M-discrete, and by (vi), $pZ \subset A$. Let $q \in A$ be a nonzero subprojection of p. Then

$$qp_\gamma = qc(p_\gamma)\, p = qc(p_\gamma) \leqslant p_\gamma,$$

thus by (i)

$$qp_\gamma = c(qp_\gamma)\, p_\gamma = c(q)\, c(p_\gamma)\, p$$

and hence by summing over γ we obtain $q = c(q)\, p$. By (i) we conclude that p is M-abelian.

(viii) This is an immediate consequence of the identities $(A_q)_p = A_p$ and $Z_p = (Z_q)_p = (Z_{M_q})_p$.

Notice that by (i) every M-abelian projection is minimal among the projections of a with the same central support (in Z) and every such projection is M-abelian if $Z \subset A$. By (vi) if A is M-discrete then $Z \subset A$.

The following theorem is modelled on the canonical decomposition of a von Neumann algebra into a continuous and a discrete part:

THEOREM 9. *There is a unique decomposition of the identity into two mutually orthogonal projections e_d, $e_c \in Z_A$ such that e_d is M-discrete and e_c is M-continuous.*

Proof. Let $e_d = \sup\{p \in A \mid p\ M\text{-abelian projection}\}$ and let $e_c = I - e_d$. By definition, e_c is M-continuous, by Lemma 8(iv), e_d is M-discrete, and by Lemma 8(v), e_d majorizes any other M-discrete projection. Let u be a unitary operator in A, then by Lemma 8(iii), ue_du^* is M-discrete and hence $ue_du^* \leqslant e_d$. Similarly $u^*e_du \leqslant e_d$ and hence e_d commutes with every unitary operator in A and thus belongs to Z_A. Q.E.D.

In the case when $Z \subset A$, our definition of M-abelian projection coincides with Guichardet's definition of relative minimal projection in [1] and some of the results collected in Lemma 8 and the following remark are contained in Lemmas 3, 4, 5, 7, 8 and Propositions 1, 2, 3, 8 in [1]. Furthermore, if A is abelian, Theorem 9 coincides with Proposition 7 in [1], while in the general case the decomposition given by Guichardet in [1, Propositions 5 and 6] is different from ours.

For a further analysis of the properties of M-continuous and M-discrete projections, we shall need the notion of n-partitions:

DEFINITION 10. Let p be a projection in A and let n be a cardinal number.

An n-partition of p (relative to M) is a decomposition of p into the sum of n mutually orthogonal projections $p_k \in A$ with $c(p_k) = c(p)$.

An n-abelian partition of p is an n-partition of p such that the projections p_k are all M-abelian.

An n-equipartition of p is an n-partition of p such that the projections p_k are all equivalent (in M).

When we do not need to specify the cardinality of the partition we shall drop n and if necessary we shall add M to the notations (e.g., M-abelian partition). Similarly, if we do not need to distinguish between different infinite cardinals, we shall write ∞-partition, etc.

Notice that for any $m < n$ we can obtain an m-partition from an n-partition by summing together projections in the n-partition. Similarly, if m divides n, we can obtain an m-equipartition from an n-equipartition. In particular, we can obtain m-equipartitions for every $m<\infty$ from an ∞-equipartition.

We can now characterize M-abelian projections in terms of 2-partitions:

LEMMA 11. *Let $p \in A$ be a projection such that $pZ \subset A$, then the following conditions are equivalent*:

(i) *p is M-abelian.*

(ii) *If $q \in A$ is a projection and $0 \neq q \leqslant p$, then q has no 2-partitions.*

(iii) *If $f \in Z$ is a projection and $0 \neq f \leqslant c(p)$, then pf has no 2-partitions.*

Proof. (i)$\Rightarrow$(ii). Assume that q has a decomposition into two mutually orthogonal projections q_1, $q_2 \in A$, then by Lemma 8(i) we have $q_i = c(q_i)p$ for $i=1, 2$. Therefore the projections $c(q_i)$ are mutually orthogonal, and hence $\{q_1, q_2\}$ cannot be a 2-partition of q.

(ii)$\Rightarrow$(iii). Obvious.

(iii)$\Rightarrow$(i). Reasoning by contradiction and applying Lemma 8(i), assume that for some projection $q \in A$, $q \leqslant p$, we have $r = pc(q) - q \neq 0$. As $pZ \subset A$, we see that also $qZ \subset A$, therefore $pc(q)$, $pc(r)$, $qc(r)$, and hence r belong to A. Since $c(r) \leqslant c(q) \leqslant c(p)$, we see that $c(qc(r)) = c(r)$, so that $pc(r)$ has the 2-decomposition $pc(r) = qc(r) + r$. Q.E.D.

If the center Z of M is contained in A, then we have a "halving" property for M-continuous projections. More generally:

PROPOSITION 12. *Let $p \in A$ be an M-continuous projection such that $pZ \subset A$, then p has a 2-decomposition.*

Proof. Let $\{e_\lambda\}$ be a maximal family of nonzero mutually orthogonal central subprojections of $c(p)$ such that pe_λ has a 2-decomposition $pe_\lambda = p_{\lambda,1} + p_{\lambda,2}$, and let $e = \sum_\lambda e_\lambda$, $p_1 = \sum_\lambda p_{\lambda,1}$, and $p_2 = \sum_\lambda p_{\lambda,2}$. Then pe has the 2-decomposition $pe = p_1 + p_2$. Let f be any central subprojection of $c(p) - e$, then by the maximality of the family $\{e_\lambda\}$, pf has no nonzero 2-decomposition. But then, $p(c(p) - e)$ is M-abelian by Lemma 11, and since it is a subprojection of the M-continuous projection p, it is M-continuous as well, and hence it is zero. Thus we conclude that $c(p) = e$ and hence that $p = p_1 + p_2$. Q.E.D.

Since subprojections (in A) of p satisfy the same hypotheses as p, we can iterate this construction and find an infinite sequence of mutually orthogonal projections $p_n \leqslant p$ with $c(p_n) = c(p)$. By adding if necessary to p_1 the complement in p of the sum of the projections p_n, we can further assume that $p=\sum_{n=1}^{\infty} p_n$. Thus p has an ∞-partition, and for any integer m we can regroup it into an m-partition:

COROLLARY 13. *Let $p \in A$ be an M-continuous projection such that $pZ \subset A$ and let $m \leqslant \aleph_0$, then p has an m-partition.*

The condition that $pZ \subset A$ is essential in Proposition 12 and Corollary 13 as we see by considering the case when $A = B(H) \otimes I$ and $M = B(H) \otimes C$ for some abelian algebra $C \neq \mathbb{C}I$. Indeed, for any projection $0 \neq p \in B(H)$ we see that $p \otimes I Z = p \otimes C$ is not contained in A, whence we conclude also that A has no nonzero M-abelian projections and hence is M-continuous. On the other hand, if p is a rank-one projection in $B(H)$, there are no decompositions of $p \otimes I$ into two mutually orthogonal nonzero projections of A. In particular, $p \otimes I$ cannot have a 2-decomposition.

4. RELATIVE DISCRETE PROJECTIONS

As in the intrinsic type decomposition theory of a von Neumann algebra, the following lemma is the key to the analysis of the structure of M-discrete projections.

LEMMA 14. *Let $p \in A$ be an M-discrete projection, then there is an M-abelian subprojection q of p such that $c(q) = c(p)$.*

Proof. Let $\{q_\lambda\}$ be a maximal family of nonzero M-abelian subprojections of p with mutually orthogonal central supports $c(q_\lambda)$ and let $q = \sum_\lambda q_\lambda$; then q is an M-abelian projection by Lemma 8(vii). Assume by contradiction that $f = c(p) - c(q) \neq 0$. By Lemma 8(v), p is the sum of mutually orthogonal M-abelian projections p_γ, thus there is an index γ such that $p_\gamma f \neq 0$. But $p_\gamma f$ is M-abelian by Lemma 8(i) and (ii), against the maximality of the family $\{q_\lambda\}$. Q.E.D.

Notice that the condition in Lemma 14 is in general not sufficient to guarantee that p be M-discrete: consider for instance a masa A in $M = B(H)$ that has both a continuous and an atomic part. The only projections that are M-discrete are those in the atomic part of A, thus I is not M-discrete, but it majorizes minimal (hence M-abelian) projections with central support I.

Now we decompose centrally M-discrete projections into "homogeneous" parts:

THEOREM 15. *Let $p \in A$ be an M-discrete nonzero projection, then there is a decomposition of $c(p)$ into mutually orthogonal nonzero central projec-*

tions f_n indexed by a set $N(p)$ of cardinal numbers, such that pf_n has an n-abelian partition.

Proof. Let $\{g_\gamma\}$ be a maximal family of mutually orthogonal central subprojections of $c(p)$ such that for every γ, pg_γ has an n-abelian partition for some positive integer or infinite cardinal n. Let $g=\sum_\gamma g_\gamma$ and let $f=c(p)-g$. Assume by contradiction that $f\neq 0$, so that $pf\neq 0$. Let q be the sum of a maximal family of mutually orthogonal nonzero M-abelian projections $q_\lambda \leqslant pf$ such that $c(q_\lambda)=f$ and let n be the cardinality of the family. Since $pf-q$ is M-discrete, by Lemma 14 there is an M-abelian projection $q_0 \leqslant pf-q$ such that $c(q_0)=c(pf-q)$. By the maximality of the family $\{q_\lambda\}$, we conclude that $c(q_0)\neq f$ and hence $g_0=f-c(pf-q)\neq 0$. But then

$$pg_0=qg_0=\sum_\lambda q_\lambda g_0$$

and each $q_\lambda g_0$ belongs to A, is M-abelian because so is q_λ, and has central support g_0. Thus pg_0 has an n-abelian partition, against the maximality of the family $\{g_\gamma\}$. Therefore we conclude that $c(p)=g$. Now let f_n be the sum of the projections g_γ for which the partition of pg_γ has the same cardinality n. By Lemma 8(vii), we can reassemble the n-abelian partitions of pg_γ into an n-abelian partition of pf_n, which concludes the proof. Q.E.D.

COROLLARY 16. *An M-discrete masa of M is unique up to inner automorphisms of M.*

Proof. Let A be an M-discrete masa of M. By Lemma 7, M-abelian projections are abelian relative to M, hence M is of type I, and we can assume, by passing if necessary to central summands of M, that M is homogeneous of type I_m for some finite integer or infinite cardinal m. By Theorem 15 applied to $p=I$, there is a decomposition of the identity into mutually orthogonal central projections $\{f_n\}_{n\in N(I)}$ such that f_n has an n-abelian partition, which, again by Lemma 7, is also an n-equipartition. Thus $f_n=0$ for $n\neq m$ and hence $f_m=I$. Let $\{p_k\}_{k\in K}$ be an m-abelian partition of I (with card $K=m$), then

$$A=\sum_{k\in K}\oplus A_{p_k}=\sum_{k\in K}\oplus Z_{p_k}.$$

If B is another M-discrete masa of M, we similarly have

$$B=\sum_{k\in K}\oplus Z_{q_k},$$

for some m-abelian partition $\{q_k\}$ of I. But since the projections p_k, q_k are abelian relative to M and share the same central support (the identity), they are equivalent (in M), so that we can find a unitary operator $u \in M$ such that $up_k u^* = q_k$. It is then clear from the above representations of A and B that ad u maps A onto B. Q.E.D.

Remark 17. (i) If M is a finite type I algebra and $Z \subset A$, then by Corollary 13, A cannot contain nonzero M-continuous projections and hence it is M-discrete.

(ii) As a consequence of (i) and Corollary 16 we see that masas of finite type I algebras are unique up to inner automorphisms (cf. [3, Lemma 3.7]).

The method in the proof of Corollary 16 can be used to obtain the following technical lemma:

LEMMA 18. (i) *Let $\{p_k\}_{k \in K}$ be a decomposition of the identity into mutually orthogonal M-abelian projections and let $C = \sum_{k \in K} \oplus Z_{p_k}$. Then C is an A-discrete masa of A.*

(ii) *If A is an M-discrete algebra, every decomposition of the identity into mutually orthogonal projections of A is contained in some A-discrete masa of A.*

Proof. C is clearly an abelian algebra, and since $A_{p_k} = Z_{p_k}$ for each k, we see that $C \subset A$. Let $x \in A \cap C'$, then x commutes with each p_k so that $x \in \sum_{k \in K} \oplus A_{p_k} = C$. Thus C is a masa of A. Moreover, $C_{p_k} = Z_{p_k} = (Z_A)_{p_k}$, whence we see that each p_k is abelian also relative to the embedding of C into A. Thus C is A-discrete.

(ii) Let $\{p_k\}_{k \in K}$ be a decomposition of the identity into mutually orthogonal projections. By Lemma 8(v), each projection p_k has a decomposition into mutually orthogonal M-abelian projections $p_k = \sum_{j \in J(k)} p_{k,j}$. Let $C = \sum_{k \in K} \sum_{j \in J(k)} \oplus Z_{p_{k,j}}$, then C is an A-discrete masa of A by (i), and $p_k \in C$ for each k. Q.E.D.

PROPOSITION 19. *Let $p \in A$ be an M-discrete projection and assume that there are two M-abelian partitions of p with cardinality n and m. Then $m = n$.*

Proof. By Lemma 8(viii), by passing if necessary to the embedding of A_p into M_p, we can assume for ease of notations that $p = I$. Thus assume that $\{p_k\}_{k \in K}$, $\{q_j\}_{j \in J}$ are two M-abelian partitions of I of cardinality n and m, respectively. Let

$$C_1 = \sum_{k \in K} \oplus Z_{p_k} \quad \text{and} \quad C_2 = \sum_{j \in J} \oplus Z_{q_j}.$$

By Lemma 18(i), C_1 and C_2 are A-discrete masas of A, hence we can apply Corollary 16 to the embedding of C_1 and C_2 into A and find an inner automorphism of A mapping C_1 onto C_2. By Lemma 8(ii), the image under such an automorphism of an M-abelian projection is M-abelian and it has the same central support, hence the image of an n-abelian partition is also an n-abelian partition. Thus, to simplify the notations, we can assume that $C_1 = C_2$. But then, for every $k \in K$ we have $p_k \in C_2$, and hence $p_k = \sum_{j \in J} f_{k,j}\, q_j$ for some elements $f_{k,j} \in Z$. Since $c(q_j) = I$ for all j, a routine argument shows that all the elements $f_{k,j}$ are projections and are uniquely determined. Since $f_{k,j} q_j \leqslant p_k$, and p_k is M-abelian, by Lemma 8(i) we have for all k, j that

$$f_{k,j} q_j = c(f_{k,j} q_j)\, p_k = f_{k,j} c(q_j)\, p_k = f_{k,j}\, p_k.$$

Thus the projections $\{f_{k,j}\}_{k \in K}$ (resp. $\{f_{k,j}\}_{j \in J}$) are mutually orthogonal for all $j \in J$ (resp. for all $k \in K$). By summing over J, we also get for every $k \in K$ that $p_k = \sum_{j \in J} f_{k,j} p_k$. Since $c(p_k) = I$, we conclude that $\sum_{j \in J} f_{k,j} = I$ for every k. Similarly, $\sum_{k \in K} f_{k,j} = I$ for every j. Now we can prove that $n = m$. Assume without loss of generality that $n \leqslant m$. If $n < \infty$, then by summing over K and interchanging the order of summation we obtain

$$nI = \sum_{k \in K} \sum_{j \in J} f_{k,j} = \sum_{j \in J} \sum_{k \in K} f_{k,j} = \sum_{j \in J} I$$

whence we conclude that $m = n$. If n and hence m are infinite, then we choose a normal state φ on M and let $a_{k,j} = \varphi(f_{k,j})$. Thus $0 \leqslant a_{k,j} \leqslant 1$ and by the normality of φ we have

$$\sum_{k \in K} a_{k,j} = \sum_{j \in J} a_{k,j} = 1 \qquad \text{for all } k \text{ and all } j.$$

Let $L = \{(k,j) \mid a_{k,j} \neq 0\}$; then $L = \bigcup_{k \in K} \{(k,j) \mid a_{k,j} \neq 0\}$. Since the series $\sum_{j \in J} a_{k,j}$ converges for every k, $\operatorname{card}\{j \mid a_{k,j} \neq 0\} \leqslant \aleph_0$ for every k and hence

$$\operatorname{card} L \leqslant \operatorname{card} K \aleph_0 = n \aleph_0 = n.$$

On the other hand, $\sum_{k \in K} a_{k,j} \neq 0$ for every j, and hence $\operatorname{card} L \geqslant \operatorname{card} J = m$, whence we obtain that $m = n$ in the infinite case too. Q.E.D.

Remark 20. As a consequence of Proposition 19 the decomposition $c(p) = \sum_{n \in N(p)} f_n$ given in Theorem 15 for an M-discrete projection p is canonical, and so is the index set $N(p)$ of degrees of M-homogeneity of p. If $p = I$, we shall call A_{f_n} the M-homogeneous summand of type I_n of A.

COROLLARY 21. *If A is an M-homogeneous algebra of degree n and a projection $0 \neq p \in A$ has an m-partition, then $m \leqslant n$.*

Proof. Let $p = \sum_{k \in K} p_k$ be an m-partition of p. By Lemma 14 we can find M-abelian projections $q_k \leqslant p_k$ such that $c(q_k) = c(p_k) = c(p)$. Then $q = \sum_{k \in K} q_k$ has an m-abelian partition. If $q = I$, then $m = n$ by Proposition 18. If $q \neq I$, then by Theorem 15 applied to the M-discrete projection $I - q$, we can find a nonzero central projection f and a positive integer or infinite cardinal m' such that $(I-q)f$ has an m'-abelian partition. Then f has both an $m+m'$-abelian partition and an n-abelian partition inherited from I, thus, again by Proposition 18, we have $m + m' = n$. In either case we see that $m \leqslant n$. Q.E.D.

If A is an M-discrete algebra then we can list the collection of all the projections of A as follows: let $\{f_n\}_{n \in N(I)}$ be the canonical decomposition of the identity into nonzero mutually orthogonal central projections f_n with n-abelian partitions $\{p_{n,k}\}_{k \in K_n}$. In Proposition 19 and its proof we have seen that n-abelian partitions of the canonical projections f_n are unique up to inner automorphisms of A and "central permutations." Thus, we define

$$P(A) = \left\{ u \left(\sum_{n \in N(I)} \sum_{k \in K_n} f_{n,k} p_{n,k} \right) u^* \mid f_{n,k} \in Z \text{ projection, } u \in A \text{ unitary} \right\}.$$

COROLLARY 22. *$P(A)$ is the collection of all the projections of the M-discrete algebra A.*

Proof. Clearly, all the elements of $P(A)$ are projections and since $p_{n,k} Z \subset A$ by Lemma 8(i), we see that $P(A) \subset A$. Conversely, if p is a nonzero projection in A, then by Lemma 18(ii) there is an A-discrete masa C of A containing p. By the same lemma

$$\sum_{n \in N(I)} \sum_{k \in K_n} \oplus Z_{p_{n,k}}$$

too is an A-discrete masa of A. Thus by Corollary 16, there is a unitary operator $u \in A$ such that

$$u^* C u = \sum_{n \in N(I)} \sum_{k \in K_n} \oplus Z_{p_{n,k}}.$$

As in the proof of Proposition 19, we see that

$$u^* p u = \sum_{n \in N(I)} \sum_{k \in K_n} f_{n,k} p_{n,k}$$

for a set of central projections $f_{n,k}$ (unique under the condition $f_{n,k} \leqslant c(p_{n,k}) = f_n$). Q.E.D.

5. EQUIPARTITIONS

A necessary but clearly not sufficient condition for the existence of an n-equipartition of the identity (where n is a positive integer or an infinite cardinal number) is that there exists a decomposition of the identity into n mutually orthogonal equivalent projections belonging to M, i.e., that M is isomorphic to $R \otimes B(H)$ for some von Neumann algebra R and some Hilbert space of dimension n.

In this section we are going to present a necessary and sufficient condition under the assumption that A contains the center Z of M. This assumption permits us to treat separately the cases when M is properly infinite and when it is finite and provides a halving property for M-continuous projections (Proposition 12).

The following example will illustrate the main features of the analysis in the properly infinite case.

EXAMPLE 23. Let $M = B(H)$ and let A be a subalgebra of M. We decompose A into three direct summands A_i with identity e_i, where A_1 is M-semifinite, A_2 is M-type III and M-continuous, and A_3 is M-type III and M-discrete.

Since the projections in A_1 are sums of finite projections, A_1 is the direct sum of all the type I factor summands of A with finite commutant; since the projections in A_3 are sums of minimal projections in A that are all infinite in $B(H)$, we see that A_3 is the direct sum of all the type I factor summands of A with infinite commutant. Thus A_2 is the direct sum of the type I summand of A with diffuse center and of the type II and type III summands of A.

If e_1 is infinite, then we can decompose it by using its M-semifiniteness into an infinite sum of mutually orthogonal infinite projections belonging to A, and this is an ∞-equipartition of e_1. By adding $e_2 + e_3$ to one of the projections of the partition, we thus obtain an ∞-equipartition of I. From this we can get an n-equipartition of I for every $n \leqslant \aleph_0$.

If $e_2 \neq 0$, by using its M-continuity we can find an ∞-equipartition of e_2 and hence again an n-equipartition of I for every $n \leqslant \aleph_0$.

Thus it remains to consider only the case when $e_1 + e_2$ is finite and hence e_3 is nonzero. Since H is separable and all the nonzero subprojections of e_3 are infinite, they are all equivalent to I. Thus e_3 has an n-equipartition if and only if the number m of mutually orthogonal minimal projections of A_3 is larger or equal to n. In the notations of Theorem 15 and Remark 20, m is the degree of M-homogeneity of e_3 (since M is a factor, $N(e_3)$ is a singleton).

If we identify A_3 with $\sum_{k \in K} \oplus B(H_k) \otimes I_{H_0}$ where H_k are separable

Hilbert spaces and I_{H_0} is the identity on an infinite dimensional separable Hilbert space H_0, we see that $N(e_3)=\sum_{k\in K}\dim(H_k)$.

To summarize: there is an n-equipartition of the identity if and only if either e_1+e_2 is infinite or $n\leqslant N(e_3)$. Of course, if A is a masa, then $e_3=0$, and hence I always has an n-equipartition.

Without the assumption of separability of H, the analysis would be considerably more complex, since infinite projections are no longer necessarily equivalent. To avoid these (nontrivial) complications we shall assume in the properly infinite case that M is σ-finite (i.e., countably decomposable).

LEMMA 24. *Let $p\in A$ be a properly infinite M-semifinite projection such that $pZ\subset A$, and let $q\in M$ be a finite projection with $c(q)\leqslant c(p)$. Then there is an infinite family of mutually orthogonal finite projections $p_\lambda\in A$ with $q\lesssim p_\lambda\leqslant p$.*

Proof. Let $\{p_\omega\}$ be a maximal family of finite mutually orthogonal projections in A such that $q\lesssim p_\omega\leqslant p$. Assume by contradiction that this family is finite and let $p_0=p-\sum_\omega p_\omega$. Then p_0 is a properly infinite projection of A and $c(p_0)=c(p)$. Let s be the sum of a maximal family $\{s_\lambda\}$ of finite projections in A with mutually orthogonal central supports and such that $qc(s_\lambda)\lesssim s_\lambda\leqslant p_0$. Thus s is finite, orthogonal to each p_ω, and $qc(s)\lesssim s\leqslant p$. We shall prove that $c(s)=c(p)$, whence $qc(s)=q$, against the maximality of the family $\{p_\omega\}$. Let $f=c(p)-c(s)$, then $p_0f\in A$ and it is M-semifinite; thus we can decompose p_0f into a sum of mutually orthogonal finite projections $r_\gamma\in A$, $\gamma\in\Gamma$. Let τ be a f.s.n. trace on $M_{c(p)}$ and let f_μ be a decomposition of the identity into mutually orthogonal central projections such that $\tau(qf_\mu)<\infty$ for all μ [9]. By the maximality of the family $\{s_\lambda\}$, for every finite subset G of Γ we have $\sum_{\gamma\in G}r_\gamma\lesssim q$, and hence $\sum_{\gamma\in G}\tau(r_\gamma f_\mu)\leqslant\tau(qf_\mu)$. By the normality of the trace we thus obtain that $\tau(p_0ff_\mu)\leqslant\tau(qf_\mu)<\infty$. As p_0 is properly infinite, it follows that $p_0ff_\mu=0$ for every μ, hence $p_0f=0$, and since $f\leqslant c(p_0)$, we conclude that $f=0$. Q.E.D.

Let e_s, e_∞, e_c, and e_d be the canonical maximal projections in Z_A that are M-semifinite, M-type III, M-continuous, and M-discrete, respectively (Theorems 3 and 9).

THEOREM 25. *Let $p\in A$ be a properly infinite σ-finite projection such that $pZ\subset A$ and let $n\leqslant\aleph_0$. Let $g\leqslant c(pe_s)$ be the canonical central projection such that pe_sg is properly infinite and $pe_sg^\perp$ is finite and let $f=g^\perp c(pe_\infty e_c)^\perp$. Then there exists an n-equipartition of p if and only if either $pe_\infty e_df=0$ or $n\leqslant\min N(pe_\infty e_df)$.*

Proof. Assume first that $g \neq 0$, so that $pe_s g \sim pg$ by the σ-finiteness of p. Since $pe_s g$ is also semifinite relative to M, there is a finite projection $q \in M$ with $c(q) = c(pe_s g) = g$. Then by Lemma 24, we can find an infinite (necessarily countable) family of mutually orthogonal finite projections $r_m \in A$ with $q \lesssim r_m \leqslant pe_s g$. We decompose the sequence $\{r_m\}$ into an infinite collection of disjoint infinite subsequences $\{r_{i,k}\}$, and define $p_k = \sum_{i=1}^{\infty} r_{i,k}$. Then $c(p_k) = g$, p_k is properly infinite and σ-finite for every k, and hence $p_k \sim pg$. If we add $pg - \sum_{k=1}^{\infty} p_k$ to p_1, we still have $p_1 \sim pg$, and thus $\sum_{k=1}^{\infty} p_k = pg$ is an ∞-equipartition of pg.

Assume now that $pe_\infty e_c \neq 0$. Then $pe_\infty e_c$ is a properly infinite M-continuous projection, and $pe_\infty e_c Z \subset A$, so that we can apply Corollary 13 and find a (necessarily countable) ∞-partition $\{q_k\}$ of $pe_\infty e_c$. Since $c(q_k) = c(pe_\infty e_c)$ and q_k is properly infinite and σ-finite because $0 \neq q_k \leqslant pe_\infty$, we see again that $q_k \sim pc(pe_\infty e_c)$ for every k. If we add $pc(pe_\infty e_c) - pe_\infty e_c$ to q_1 we obtain the ∞-equipartition $\{q_k\}$ of $pc(pe_\infty e_c)$.

In the case that both $pe_s g$ and $pe_\infty e_c$ are nonzero, we can combine the two ∞-equipartitions that we found into the ∞-equipartition $\{p_k + q_k g^\perp\}$ of $pf^\perp = pg + pc(pe_\infty e_c) g^\perp$. By the remarks after Definition 10, we thus see that if nonzero, $pf^\perp$ has an n-equipartition. As a consequence, p has an n-equipartition if and only if either pf is zero or pf has an n-equipartition.

Notice first that since $f \leqslant c(pe_\infty e_c)^\perp$, we have $pf = pe_s f + pe_\infty e_d f$, $pe_s f$ is finite, and $pe_\infty e_d f$ is M-type III or zero. Therefore $pf = 0$ if and only if $pe_\infty e_d f = 0$. Assume that $pf \neq 0$ has an n-equipartition $\{p_k\}$. By the above decomposition, $p_k = p_k e_s f + p_k e_\infty e_d f$. As $p_k e_\infty e_d f \neq 0$ for some index k, we see that for that index and hence for all indices, p_k is properly infinite. As $p_k e_s f$ is finite, we see that $\{p_k e_\infty e_d f\}$ is an n-equipartition of $pe_\infty e_d f$. By Corollary 21 we conclude that $n \leqslant \min N(pe_\infty e_d f)$.

Assume on the other hand that $pf \neq 0$ and that $n \leqslant \min N(pe_\infty e_d f)$. For every $m \in N(pe_\infty e_d f)$ and the corresponding central projection f_m, the projection $pe_\infty e_d f_m$ has an m-abelian partition. By summing together enough of the projections in this partition, we obtain an n-partition of $pe_\infty e_d f_m$, and by direct summing over all m in $N(pe_\infty e_d f)$ we obtain an n-partition of $pe_\infty e_d f$. Since $pe_\infty e_d f$ is M-type III, all the projections in this partition are properly infinite, thus they are all equivalent (to pf) and hence they form an n-equipartition of $pe_\infty e_d f$. By adding $pf - pe_\infty e_d f$ to one of the projections in the partition, we thus obtain an n-equipartition of pf. Q.E.D.

Without the condition $pZ \subset A$ which enables us to use Proposition 12 and Lemma 24 and thus reduce the problem to the analysis of the M-discrete, M-type III part of A, the result as stated above would be false. Indeed if we embed $M_2(\mathbb{C})$ into $B(H)$ and we define $A = M_2(\mathbb{C}) \otimes I$ and $M = B(H) \otimes C$ for some nontrivial abelian algebra C, then reasoning as in the example after Corollary 13, we see that A is M-continuous, and hence the projection f defined in Theorem 25 vanishes. It is however obvious that A cannot have any n-partition for $n > 2$.

Let us consider now the case when M and hence A are finite algebras and let us again illustrate with an example the main features of our analysis.

EXAMPLE 26. Let H be an m-dimensional Hilbert space, $M = B(H)$ (i.e., $M = M_m(\mathbb{C})$ is the algebra of $m \times m$ matrices) and let A be a subalgebra of M. Then A is the direct sum of algebras A_k with identity e_k, where A_k is either a full matrix algebra $M_{m\,\mathrm{tr}(e_k)}(\mathbb{C})$ or it is $\mathbb{C}e_k$ (we normalize the trace tr on M so that the identity has trace 1). Let A_0 be a masa of A and let $Q(A)$ be the list of the traces of the minimal projections in A_0, then $Q(A)$ is a collection of integer multiples of $1/m$ with sum 1, and, clearly, it does not depend on the choice of the masa A_0. A moment's reflection shows that there is an n-equipartition of I for some positive integer n if and only if $Q(A)$ can be divided into n disjoint sets such that the sum of all the entries in each set is $1/n$.

A necessary condition for this to occur is of course that n divides m, i.e., that there is a decomposition of the identity into n equivalent mutually orthogonal projections of M. Notice that if A is itself a masa of M, then all the entries in $Q(A)$ are $1/m$ and hence the condition that n divides m is also sufficient. It is obvious that in general it is not.

In the general case when M is not a factor, we need to replace the scalar valued trace used in the above example with a center-valued trace, and we need to deal also with the M-continuous part of A.

Let Φ be the canonical center-valued trace on the finite algebra M. Recall that two projections p and q are equivalent in M if and only if $\Phi(p) = \Phi(q)$ [5, Theorem 8.2.8 and Proposition 8.1.1].

We start with the following Darboux property for the M-continuous part of A.

PROPOSITION 27. *Let q be an M-continuous projection of A such that $qZ \subset A$ and let $z \in Z$ be such that $0 \leqslant z \leqslant \Phi(q)$. Then there is a projection $p \in A$, with $p \leqslant q$ and $\Phi(p) = z$.*

Proof. The case when $z = 0$ is trivial. Since $qZ \subset A$, we can pass if necessary to central summands, and so we can assume that z is bounded below by a positive number α. Let

$$P = \{ p \in A \mid p \text{ projection}, p \leqslant q, \Phi(p) \leqslant z \}$$

and let $p = \sup P$. By the normality of Φ we see that $p \in P$. We claim that

$\Phi(p)=z$. Assume not. Then there is a nonzero central projection $f\leqslant c(p)$ and a number $0<\beta<1$ such that $\Phi(p)f\leqslant\beta zf$. In particular, $\Phi(p)f\neq\Phi(q)f$, hence $(q-p)f$ is nonzero, and being majorized by the M-continuous projection q, it is M-continuous itself. Let k be an integer larger than $2/(1-\beta)\alpha$, then by Corollary 13, we can find a k-partition $(q-p)f=r_1+r_2+\cdots+r_k$. If we had $\Phi(r_j)\geqslant(2/k)f$ for all j, then $\Phi((q-p)f)\geqslant 2f\geqslant 2\Phi(qf)$, whence $\Phi(qf)\leqslant 0$, against the assumption that $\Phi(qf)\geqslant zf\geqslant\alpha f$. Thus there is an $r=r_j$ and a nonzero central projection $g\leqslant c(r)=c((q-p)f)$ such that $\Phi(r)\,g\leqslant(2/k)\,g\leqslant\alpha(1-\beta)\,g$. Let $p'=p+rg$. Clearly, $p'\in A$ and $p\leqslant p'\leqslant q$. Moreover,

$$\begin{aligned}\Phi(p')&=\Phi(p)+\Phi(r)\,g\\&\leqslant\Phi(p)\,g^{\perp}+\Phi(p)\,g+\alpha(1-\beta)\,g\\&\leqslant zg^{\perp}+\beta zg+(1-\beta)\,zg\\&\leqslant z,\end{aligned}$$

so that $p'\in P$. But then $p'\leqslant p$, and hence $rg=0$, against the assumption that $g\neq 0$. Q.E.D.

Let $Q_c(A)=\{z\in Z\mid 0\leqslant z\leqslant\Phi(e_c)\}$, then as a consequence of Proposition 27 we have that

$$Q_c(A)=\Phi\{p\in A\mid p\ M\text{-continuous projection}\}.$$

We consider now the M-discrete part of A. Let $\{f_n\}_{n\in N(e_d)}$ be the canonical decomposition of $c(e_d)$ into mutually orthogonal central projections $f_n\in Z$ such that $f_n e_d$ has an n-abelian partition $\{p_{n,k}\}_{k\in K_n}$ (with card $K_n=n$). Let

$$Q_d(A)=\left\{\sum_{n\in N(e_d)}\ \sum_{k\in K_n} f_{n,k})\mid f_{n,k}\text{ projection in }Z\right\}.$$

Then by Corollary 22 applied to the embedding of A_{e_d} into M_{e_d}, we have that

$$Q_d(A)=\Phi\{p\in A\mid p\ M\text{-discrete projection}\},$$

and

$$Q(A)=Q_c(A)+Q_d(A)=\Phi\{p\in A\mid p\text{ projection}\}.$$

In particular we see that $Q_d(A)$ and hence $Q(A)$ do not depend on the choice of the n-abelian partitions $\{p_{n,k}\}$. By Proposition 19, Corollary 22 we easily verify that $Q_d(A)=Q(C)$ for any masa C of A_{e_d}.

DEFINITION 28. Let m be a positive integer. Then we say that $Q(A)$ is divisible by m if for every $n \in N(e_d)$, $k \in K_n$ there is a decomposition of the identity into m mutually orthogonal central projections $f_{j,n,k}$ and there are m central elements z_j with $0 \leqslant z_j \leqslant \Phi(e_c)$ such that

$$\sum_{j=1}^{m} z_j = \Phi(e_c)$$

and

$$z_j + \sum_{n \in N(e_d)} \sum_{k \in K_n} f_{j,n,k} \Phi(p_{n,k}) = \frac{1}{m} I \qquad \text{for every} \quad j = 1, 2, ..., m.$$

THEOREM 29. *Let M be a finite algebra, let A contain the center Z of M, and let m be a positive integer, then the identity has an m-equipartition if and only if $Q(A)$ is divisible by m.*

Proof. Assume that $Q(A)$ is divisible by m, and let z_j and $f_{j,n,k}$ be as in Definition 28. By using Proposition 27 we can find a decomposition of e_c into m mutually orthogonal projections $r_j \in A$ with $\Phi(r_j) = z_j$. Define

$$q_j = r_j + \sum_{n \in N(e_d)} \sum_{k \in K_n} p_{n,k} f_{j,n,k}.$$

Then $q_j \in A$ is a decomposition of the identity into mutually orthogonal projections and since $\Phi(q_j) = (1/m) I$, all the q_j are equivalent (relative to M) and hence form an m-equipartition of I.

Assume on the other hand that there is an m-equipartition $\{q_j\}$ of I, so that $\Phi(q_j) = (1/m) I$, and define $z_j = \Phi(q_j e_c)$. Then $\sum_{j=1}^{m} z_j = \Phi(e_c)$. By Lemma 18(ii), there is an A-discrete masa C of A_{e_d} containing $\{q_j e_d\}$. By Corollary 16, there is a unitary operator $u \in A$ such that

$$C = \sum_{n \in N(e_d)} \sum_{k \in K_n} \oplus Z_{up_{n,k}u^*}.$$

Therefore there are unique central projections $f_{j,n,k}$ with $f_{j,n,k} \leqslant c(p_{n,k}) = f_n$, such that

$$q_j e_d = \sum_{n \in N(e_d)} \sum_{k \in K_n} up_{n,k} u^* f_{j,n,k}.$$

Using the orthogonality of the projections q_j, and the condition that $c(q_j) = I$, we see that for every $n \in N(e_d)$, $k \in K_n$ the projections $f_{j,n,k}$ form an orthogonal decomposition of the identity. This completes the proof.

Q.E.D.

Remark 30. If we need to consider equipartitions of a projection p, by Lemma 8(viii) we can pass to the embedding of A_p into M_p. We leave to the reader the details of the reduction. In particular, we see from Proposition 27 that a finite M-continuous projection p such that $pZ \subset A$, has an n-equipartition for every finite integer n.

Notice that by combining this result with the one obtained in Theorem 25, we obtain that all M-continuous σ-finite projections such that $pZ \subset A$ have n-equipartitions, which yields a stronger form of Corollary 13.

6. Applications

As a special case of Theorems 25 and 29 we can reobtain the result by Kadison [3, Theorem 3.18] on the existence of equipartitions for masas of von Neumann algebras.

Notice that if A is a masa of M, by Lemma 7 all the M-abelian projections are abelian and hence finite relative to M, so that every M-discrete projection is necessarily M-semifinite (and it is also discrete relative to M). Thus the M-type III part of A is necessarily M-continuous, i.e., $e_\infty e_d = 0$, and the M-discrete part of A is necessarily contained in the type I part of M. These inclusions permit a simpler analysis:

Corollary 31. *Assume that the properly infinite part of M is σ-finite, let A be a masa of M, and let n be a positive integer or $\aleph_0$. Then there exists an n-equipartition of the identity if and only if there exists a decomposition of the identity into n mutually orthogonal equivalent projections of M.*

Proof. The condition is obviously sufficient. Let $\{f_i\}_{i=1,2,3}$ be a decomposition of the identity into mutually orthogonal central projection such that f_1 is properly infinite, f_2 is finite and of type II, and f_3 is finite and of type I. We need to prove that each f_i has an n-equipartition. Since $e_\infty e_d = 0$ by the above remarks, we obtain that f_1 has an n-equipartition from Theorem 25. If $f_2 + f_3 \neq 0$, then by the existence of a decomposition of the identity into n mutually orthogonal equivalent projections we conclude that $n < \infty$. Since f_2 is M-continuous, it has an n-equipartition by Theorem 29, or, as a direct consequence of Proposition 27. Finally, assume that $\{p_k\}_{k=1,\ldots,n}$ is a decomposition of the identity into n mutually orthogonal equivalent projections of M and let C_0 be a masa of M_{f_3} containing $\{p_k f_3\}_{k=1,\ldots,n}$. Then by Remark 17(ii), C_0 is unitarily equivalent (in M) to A_{f_3}, so that f_3 too has an n-equipartition. Q.E.D.

We consider now an application to discrete crossed products. From now on, let A be a von Neumann algebra, G be a discrete group acting on A, and let M be the crossed product of A by G. Embed A into M, denote by E the canonical normal. faithful conditional expectation of M onto A and by $\{u_g\}_{g \in G}$ the unitary group of M that implements the action of G, so that M is generated by A and by $\{u_g\}_{g \in G}$. Then every element x of M is represented by its generalized Fourier expansion $\sum_{g \in G} x_g u_g$, where $x_g = E(xu_g{}^*) \in A$ and the series converges in the Bures topology [6; 7, 7.11].

LEMMA 32. (i) *The canonical projections e_s, e_∞, e_d, and e_c belong to the fixed point algebra Z_A^G and hence to Z.*

(ii) *A projection $p \in A$ is M-discrete if and only if there is an M-abelian subprojection q of p with $c(q) = c(p)$.*

Proof. (i) Let p be a finite projection, then $\operatorname{ad} u_g(p)$ is finite for every $g \in G$. Since e_s is the supremum of all finite projections of A, we see that e_s and hence e_∞ is invariant under the action of G. Similarly, if p is M-abelian, then

$$\begin{aligned}\operatorname{ad} u_g(p)\, A \operatorname{ad} u_g(p) &= \operatorname{ad} u_g(p \operatorname{ad} u_{g-1}(A)\, p)\\ &= \operatorname{ad} u_g(pAp)\\ &= \operatorname{ad} u_g(pZ)\\ &= \operatorname{ad} u_g(p)\, Z,\end{aligned}$$

whence $\operatorname{ad} u_g(p)$ is M-abelian for every $g \in G$. The same reasoning as above shows that e_d and hence e_c is in Z_A^G.

(ii) The necessity part is Lemma 14. Assume that q is M-abelian and that $c(q) = c(p)$. Then $q \leqslant e_d$, hence by (i) we have $c(q) \leqslant e_d$. By Lemma 8(vi) and (iii) we see that $c(p) = c(q) \in A$ and that $c(p)$ is M-discrete. Thus $p \leqslant c(p)$ is M-discrete. Q.E.D.

In particular, A is M-discrete if and only if there is an M-abelian projection with central support the identity.

LEMMA 33. (i) *Let p_1, p_2 be M-abelian projections and let $q_1, q_2 \in M$ be equivalent subprojections of p_1 and p_2, then there is an element $z \in Z$, with $0 \leqslant z \leqslant c(q_1)$, such that $E(q_1) = zp_1$ and $E(q_2) = zp_2$.*

(ii) *Let p_1, p_2 be M-abelian projections, then $p_1 \precsim p_2$ if and only if $c(p_1) \leqslant c(p_2)$.*

Proof. (i) Let $v = \sum_{g \in G} v_g u_g$ be a partial isometry in M such that $q_1 = vv^*$ and $q_2 = v^*v$. Since $\sum_{g \in G} v_g v_g^* = E(q_1)$ (where the series converges in the strong topology [7, Lemma 7.11.3]), and since $E(q_1) \leqslant E(p_1) = p_1$ by the positivity of E, we have $v_g v_g^* \leqslant p_1$ for every $g \in G$. As p_1 is M-abelian, there are $z_g \in Z$ such that $v_g v_g^* = z_g p_1$. Similarly, $\sum_{g \in G} \operatorname{ad} u_g^*(v_g^* v_g) = E(q_2)$, and hence $\operatorname{ad} u_g^*(v_g^* v_g) = w_g p_2$ for some $w_g \in Z$.

Now $c(q_i) \leqslant c(p_i) \leqslant c(e_d) = e_d$ for $i = 1, 2$, by Lemma 32(i), and hence $c(q_i) \in A$ by Lemma 8(vi). Thus $E(q_i) = c(q_i)\, E(q_i)$ and hence we can choose the elements z_g and w_g so that $0 \leqslant z_g \leqslant c(q_1)$, $0 \leqslant w_g \leqslant c(q_2)$. For every central projection $e \leqslant c(q_1) = c(q_2)$, we have

$$\|v_g v_g^* e\| = \|z_g p_1 e\| = \|z_g e\|$$

and similarly

$$\|\mathrm{ad}\, u_g^*(v_g^* v_g)\, e\| = \|w_g e\|,$$

but since

$$\|\mathrm{ad}\, u_g^*(v_g^* v_g)\, e\| = \|\mathrm{ad}\, u_g^*(v_g^* v_g e)\| = \|v_g^* v_g e\| = \|v_g v_g^* e\|,$$

we conclude that $\|z_g e\| = \|w_g e\|$ and hence $z_g = w_g$. Thus $z = \sum_{g \in G} z_g$ satisfies the required condition.

(ii) The condition is clearly necessary, and to prove its sufficiency it is enough to show that $c(p_1) = c(p_2)$ implies $p_1 \sim p_2$. By the Comparison Theorem there is a central projection e and projections q_1, $q_2 \in M$, $q_1 \leqslant ep_1$, and $q_2 \leqslant e^\perp p_2$ such that $ep_2 \sim q_1$ and $e^\perp p_1 \sim q_2$. By (i) there is a central element $0 \leqslant z \leqslant c(ep_2)$ such that

$$E(ep_2) = zep_2 \qquad \text{and} \qquad E(q_1) = zep_1.$$

Since $ep_2 \in A$ by Lemma 8(vi), we see that $z = c(ep_2) = ec(p_1)$ and hence $E(q_1) = ep_1$. But then $E(ep_1 - q_1) = 0$, and since E is faithful, we obtain that $q_1 = ep_1$. Similarly, $q_2 = e^\perp p_2$, and hence $p_1 \sim p_2$. Q.E.D.

To complete our results for the discrete crossed product embedding, we need the following lemma, which we prove in the general case. We shall return in another paper to the analysis of the relations between semifiniteness of M, of A, of the embedding of A into M (i.e., M-semifiniteness of A), the existence of semifinite traces, and related issues.

LEMMA 34. *If A is an M-semifinite subalgebra of M and if $Z \subset A$, then every faithful, semifinite, normal (f.s.n. for short) trace τ on M has a f.s.n. restriction to A.*

Proof. Notice that the M-semifiniteness of A implies the semifiniteness of M. Let τ be a f.s.n. trace on M. Then clearly τ is a f.n. trace on A, and it remains only to prove that it is semifinite. By Lemma 2(ii) there is a decomposition of the identity into mutually orthogonal finite projections $p_\gamma \in A$, and by [9], for each γ there is a decomposition of the identity into mutually orthogonal central projections $\{f_{\gamma,\mu}\}$ such that $\tau(p_\gamma f_{\gamma,\mu}) < \infty$ for each μ. By hypothesis, $p_\gamma f_{\gamma,\mu} \in A$, and by construction $\sum_{\gamma,\mu} p_\gamma f_{\gamma,\mu} = I$. Therefore the restriction of τ to A is semifinite. Q.E.D.

We return now to the case where M is a crossed product of A by the discrete group G.

PROPOSITION 35. (i) *Every M-abelian projection is finite.*

(ii) *Every M-discrete projection is M-semifinite.*

(iii) *If A is M-discrete, then there exists a f.s.n. G-invariant trace on A.*

Proof. (i) If p is an M-abelian projection and it is equivalent to a subprojection $q \in M$, then from Lemma 33(i) we see that $E(q) = zp$ for some $z \in Z$ with $0 \leqslant z \leqslant c(p)$ and that $p = E(p) = zp$. Thus $z = c(p)$ and hence $E(p - q) = 0$, whence $p = q$.

(ii) Obvious from (i) and Lemma 8(v).

(iii) Obvious from (ii) and Lemma 34. Q.E.D.

A similar result was obtained by Størmer [11, Lemma 9] for the case of $\sim_G$-abelian projections. The notions of M-abelian and $\sim_G$-abelian projections coincide when $Z = Z_A^G$ (which happens if and only if $Z \subset A$, e.g. when A is M-discrete), but they are different in general. The technique used in Lemma 33 and Proposition 35 can be carried over to obtain the analogous results for the case of $\sim_G$-abelian projections.

COROLLARY 36. *Every properly infinite σ-finite projection $p \in A$ such that $pZ \subset A$ has an ∞-equipartition.*

Proof. This is an immediate consequence of Theorem 25 and the fact that the M-discrete projection $pe_\infty e_d$ is at the same time M-type III by definition and M-semifinite by Proposition 35(ii) and hence it is zero. Q.E.D.

This result answers (for the case that $Z \subset A$) a question by Pedersen and Størmer. In [8] they investigated an equivalence relation among projections of A which was defined in [11] as an extension of the Hopf equivalence relation in ergodic theory. In [8, Sect. 3] they proved that this equivalence relation agrees with the one inherited from the Murray–von Neumann equivalence in M, and that the class of finite elements under it coincides with the class of finite projections of A (relative to M).

In [8, Theorem 4.2] they characterized this class by proving that (in our notations) a σ-finite projection $p \in A$ is finite (relative to M) if and only if for some (and hence any) normal state ω of A with support p and $\varepsilon > 0$ there is $\delta > 0$ such that $\omega(x) < \delta$ and $x \approx y$ implies $\omega(y) < \varepsilon$ for all elements x, y in A^+ majorized by p (where $\approx$ is the Kadison–Pedersen equivalence relation relative to M [4, 8]).

Their question was whether asking for x, y to be just subprojections of

p was enough, and they proved that this was the case when A was abelian [8, Theorem 5.3] using a technique developed by Singer [10].

An inspection of the proof of their Theorem 4.2 shows that the key step that permits one to extend it to projections is (again in our notations) the existence of ∞-equipartitions for properly infinite projections. This is given now by Corollary 36. Thus we have:

COROLLARY 37. *Let $p \in A$ be σ-finite projection such that $pZ \subset A$, then p is finite if and only if for some (and hence any) normal state ω of A with support p and $\varepsilon > 0$ there is $\delta > 0$ such that $\omega(r) < \delta$ and $r \sim q$ implies $\omega(q) < \varepsilon$ for all subprojections r, $q \in A$ of p.*

REFERENCES

1. A. GUICHARDET, Une caractérization des algèbres de von Neumann discrètes, *Bull. Soc. Math. France* **89** (1961), 77–101.
2. R. V. KADISON, Normalcy in operator algebras, *Duke Math. J.* **29** (1962), 459–464.
3. R. V. KADISON, Diagonalizing matrices, *Amer. J. Math.* **106** (1984), 1451–1468.
4. R. V. KADISON AND G. K. PEDERSEN, Equivalence in operators algebras, *Math. Scand.* **2** (1970), 205–222.
5. R. V. KADISON AND J. R. RINGROSE, "Fundamentals of the Theory of Operator Algebras," Vol. II, Academic Press, New York, 1984.
6. R. MERCER, Convergence of Fourier series in discrete cross products of von Neumann algebras, *Proc. Amer. Math. Soc.* **94** (1985), 254–258.
7. G. K. PEDERSEN, "C^*-Algebras and Their Automorphism Groups," Academic Press, London, 1979.
8. G. K. PEDERSEN AND E. STØRMER, Automorphisms and equivalence in von Neumann algebras, II, *Indiana Univ. Math. J.* **23** (1973), 121–129.
9. C. PELIGRAD AND L. ZSIDÓ, A Riesz decomposition theorem in W^*-algebras, *Acta Sci. Math.* **34** (1973), 317–323.
10. I. M. SINGER, Automorphisms of finite factors, *Amer. J. Math.* **77** (1955), 117–133.
11. E. STØRMER, Automorphisms and equivalence in von Neumann algebras, *Pacific J. Math.* **44** (1973), 371–383.